Stat 230

Homework Problem Set 4

Due Jun 5 12:30 pm EST.

Please justify your answers and show the steps that lead you to your answer. Without a proper explanation, even a perfectly correct answer will receive a low score. Doubts regarding the problem sets/ Problem setup can be posted on Piazza. Simplify the expressions whenever you can.

Pr. 1 Suppose that $X_1, \ldots, X_n$ form a random sample of size n from the uniform distribution on the interval $[0, 1]$, and that $Y_n = \max\{X_1, \ldots, X_n\}$. Find the smallest value of n such that

$$\Pr(Y_n \geq 0.99) \geq 0.95.$$

Pr. 2 An electronic device has lifetime denoted by T. The device has value $V = 5$ if it fails before $t = 3$, otherwise, it has value $V = 2T$. Find the cdf of V, if T has pdf

$$f_T(t) = \frac{1}{1.5} e^{-t/(1.5)}, \ t > 0.$$

Pr. 3 Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$ for each of the following probability distributions

a) $f_X(x) = ax^{a-1}, \ 0 < x < 1, \ a > 0$

b) $f_X(x) = \frac{1}{n}, x = 1, 2, \ldots, n, \ n > 0$ a fixed integer.

c) $f_X(x) = \frac{3}{2}(x-1)^2, \ 0 < x < 2$

Pr. 4 Suppse X is a random variable such that its moment generating function is:

$$\psi_X(t) = \frac{1}{10}\Big(4 + e^{-t} + 2e^{5t} + 3e^{-5t}\Big), \quad -\infty < t < \infty.$$

What is the distribution of X? Justify your answer.

Pr. 5 Suppose that an electronic system contains 20 similar components that work independently of each other, and are connected in series such that the whole system fails as soon as any of the components fails. Suppose the lifetime of each component has the exponential distribution with mean of 50,000 hours.

a) Determine the mean and the variance of the length of the time until the system fails

b) What is the probability that the system lasts longer than 2,000 hours?

c) An engineer adds a fail protection to the system, so that it can keep running with 19 components working (so it will break once the second component fails). How does this fail protection system change the answer to b)?

Pr. 6 It is said that a random variable X has the Pareto distribution with parameters x_0 and α ($x_0 > 0$ and $\alpha > 0$) if X has a continuous distribution for which the p.d.f. $f_X(x|x_0, \alpha)$ is as follows:

$$f_X(x) = \begin{cases} \frac{\alpha x_0^\alpha}{x^{\alpha+1}}, & \text{for } x \geq x_0, \\ 0, & \text{for } x < x_0. \end{cases}$$

Show that if X has this Pareto distribution, then the random variable $Y = \log(X/x_0)$ has the exponential distribution with parameter α.