# Stat 230 Summer 2020 I

## Notes on the moment generating function

For $X$ a random variable, the moment generating function (mgf) is defined as:

$$\psi_X(t) = \mathbb{E}[e^{tX}] = \mathbb{E}[1 + tX + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \frac{(tX)^4}{4!} + \dots],\ t \in (-\infty, \infty),$$

i.e. we associate to $X$ a function $\psi_X : \mathbb{R} \to \mathbb{R}, t \mapsto \mathbb{E}[e^{tX}]$.

The mgf may or may not be finite at any particular value of $t$, but the Taylor expansion above tells us it will be 1 at $t = 0$. If the mgf is bounded around some interval $[-\delta, \delta]$ of 0, then the mgf uniquely determines the law of $X$.

**Theorem 0.1.** *If the moment generating function $\psi_X(t), \psi_Y(t)$ of two random variables $X$ and $Y$, respectively, are finite and equal around any open interval around 0, then the law of $X$ is the law of $Y$.*

This important theorem allows us to represent many familiar distributions in terms of other distributions. The following Theorem is used in conjunction with Theorem 0.1 to show how certain distributions arise as transformations of some other distributions.

**Theorem 0.2.** *The following properties are true for the moment generating function $\psi(t)$:*

1) $\psi_X(0) = 1$
2) $\psi_{aX+b} = e^{bt}\psi(at), t \in \mathbb{R}$.
3) *If $X$ and $Y$ are independent, then $\psi_{X+Y}(t) = \psi_X(t)\psi_Y(t), t \in \mathbb{R}$.*
4) *If $\psi_X(t)$ is finite in an open interval around zero, then it is infinitely differentiable at zero.*

First thing to note about this theorem is that the mgf of $X$ allows us to compute moments of $X$ (hence the name)

**Corollary 0.3.** *If the moment generating function $\psi_X(t)$ of $X$ is finite in an open interval around zero, then*

$$\frac{\partial^k \psi_X(t)}{\partial t^k}\Big|_{t=0} \psi_X(t) = \mathbb{E}[X^k].$$

*Proof.* Expectation is a linear operator and differentiable power series can be differentiated by terms.

$$\begin{aligned}
\frac{\partial^k \psi_X(t)}{\partial t^k}\Big|_{t=0} \psi_X(t) &= \frac{\partial^k}{\partial t^k}\Big|_{t=0} \mathbb{E}\Big[1 + Xt + \frac{(Xt)^2}{2!} + \dots + \frac{(Xt)^{k-1}}{(k-1)!} + \frac{(Xt)^k}{(k)!} + \frac{(Xt)^{k+1}}{(k+1)!} + \dots\Big] \\
&= \mathbb{E}\Big[\frac{\partial^k}{\partial t^k}\Big|_{t=0}\Big(\underbrace{1 + Xt + \frac{(Xt)^2}{2!} + \dots + \frac{(Xt)^{k-1}}{(k-1)!}}_{\text{killed by the kth derivative}} + \frac{(Xt)^k}{(k)!} \underbrace{+ \frac{(Xt)^{k+1}}{(k+1)!} + \dots}_{\text{killed by evaluation at t=0}}\Big)\Big] \\
&= \mathbb{E}\Big[\frac{\partial^k \psi_X(t)}{\partial t^k}\Big|_{t=0} \frac{(Xt)^k}{(k)!}\Big] = \mathbb{E}\big[X^k\big].
\end{aligned}$$

□

The following example illustrates a typical use case on how the moment generating function can be used to derive relations between distributions.

**Example 0.4.** *Let $X_1, \dots, X_n$ be independent and identically distributed random variables with distribution exp($\lambda$). Then $Y = \sum_{i=1}^n X_i$ has the Gamma($n, \lambda$) distribution.*

*Firstly, the moment generating function of $X_i$ is, for $t < \lambda$:*

$$\begin{aligned}\psi_{X_i}(t) = \mathbb{E}[e^{tX_i}] &= \int_{x=0}^{\infty} e^{tx} \lambda e^{-\lambda x} \, dx \\ &= \frac{\lambda}{\lambda - t} \int_{x=0}^{\infty} (\lambda - t) e^{-(\lambda - t)x} \, dx \\ &= \frac{\lambda}{\lambda - t}.\end{aligned}$$

*The moment generating function of $Y$ is, for $t < \lambda$:*

$$\begin{aligned}\psi_Y(t) = \mathbb{E}[e^{tY}] &= \int_{x=0}^{\infty} e^{ty} \frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-\lambda y} \, dy \\ &= \frac{\lambda^n}{(\lambda - t)^n} \int_{x=0}^{\infty} \frac{(\lambda - t)^n}{\Gamma(n)} y^{n-1} e^{-(\lambda - t)y} \, dy \\ &= \frac{\lambda^n}{(\lambda - t)^n}.\end{aligned}$$

*On the last line we used a common trick in statistics: The integrand is density of a Gamma($n, (\lambda - t)$) random variable. Hence it integrates to 1.*

*Finally, by using Theorem 0.2 we see that*

$$\begin{aligned}\psi_{\sum_{i=1}^n X_i}(t) &= \prod_{i=1}^n \psi_{X_i}(t) \\ &= \prod_{i=1}^n \frac{\lambda}{\lambda - t} \\ &= \frac{\lambda^n}{(\lambda - t)^n}.\end{aligned}$$

*The moment generating functions of $Y$ and $\sum_{i=1}^n X_i$ are the same, therefore the law of $Y$ is that of $\sum_{i=1}^n X_i$ by Theorem 0.1. We conclude we can construct Gamma random variable[1] as a sum of iid exponentials.*

[1] with the 1st parameter being a positive integer