Midterm Practice Examination

Sta 230: Probability

June 3, 2020

This is a practice exam and not graded. The rules stated below don't apply, but should you wish to simulate the midterm, you may pretend they do.

This is an open notes open book exam, so you can consult your notes and the course books (Pitman/Durrett). You may use a calculator, R or such to evaluate complicated expressions, but you are not allowed to search the internet or ask anyone about the problems. You may express fractions as fractions (reduced to lowest terms), and report any numerical answers with 3 digits.

You must show your work to get partial credit. Even correct answers will not receive full credit without justification.

Write your solutions as clearly as possible and make sure it's easy to find your answers (circle them if necessary), since you will not receive credit for work that I cannot understand or find. Good Luck!

Problem 1 : A hand in bridge consists of thirteen cards dealt out from a well shuffled deck. A deck as 52 cards with four suits (hearts, clubs, diamonds, and spades).

a. What is the probability that the bridge hand contains exactly 5 hearts?

b. What is the probability that the bridge hand contains exactly 5 hearts and 5 spades?

c. What is the probability that the hand contains exactly 5 cards from at least one suit?

d. Given the first three cards are 3 hearts what is the probability the next two are spades ?

Problem 2 : There are two possible distributions that you are drawing data from

$$
\begin{aligned}
Y &\sim \text{Geometric}(p) \\
Y &\sim \text{Poisson}(\lambda)
\end{aligned}
$$

I run the following experiment n times identically and independently

(1) $X \sim \text{Be}(p = .3)$

(2) If X is 1 then draw $Y \sim \text{Geometric}(p)$
If X is 0 then draw $Y \sim \text{Poisson}(\lambda)$.

What is $\mathbf{E}[Y]$?

Problem 3 : $X_1, ..., X_n \overset{iid}{\sim} \mathrm{N}(\mu, \sigma^2)$

1. Define $Z = \frac{\sum_i X_i}{n}$, what is $\mathbf{E}[Z]$ and $\mathbf{V}[Z]$?
2. How do I compute transform the random variable Z into a standard normal variable W ?
3. Now suppose that $X_1, ..., X_n \overset{iid}{\sim} \mathrm{Bin}(m, p)$ again define $Z = \frac{\sum_i X_i}{n}$. For what value of m, p would Z approximate the distribution in part (1) ?

Problem 4:

1. Let E,F and G be events. Prove or give counterexamples to the following statements:

   a) If E is independent of F and E is independent of G, then E is independent of $F \cup G$.

   b) If E is independent of F, and E is independent of G, and $F \cap G = \emptyset$, then E is independent of $F \cup G$.

   c) If E is independent of F, and F is independent of G, and E is independent of $F \cap G$, then G is independent of $E \cap F$.

2. Let A and B be events having positive probability. State whether each of the following statements is (i) necessarily true, (ii) necessarily false, or (iii) possibly true.

   a) If A and B are mutually exclusive, then they are independent.

   b) If A and B are independent, then they are mutually exclusive.

   c) $\Pr(A) = \Pr(B) = 0.6$, and A and B are mutually exclusive.

   d) $\Pr(A) = \Pr(B) = 0.6$, and A and B are independent.

Problem 5 : . A new drug for leukemia works 25% of the time in patients 55 and older, and 50% of the time in patients younger than 55. A test group consists of 17 patients 55 and older and 12 patients younger than 55.

1. A patient is chosen uniformly at random from the test group, the drug is administered, and it is a success. What is the probability the patient was in the older group?
2. A subgroup of 4 patients are chosen and the drug is administered to each. What is the probability that the drug works in all four patients?

Problem 6 : . Suppose that the cumulative distribution function of the random variable X is given by

$$F_X(x) = 1 - e^{-x^2}, \quad x > 0$$

Find

a) $\Pr(X > 2)$

b) $\Pr(1 < X < 3)$

c) $\mathsf{E}[X]$

d) $\mathrm{Var}(X)$

Name: ____________________

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt:$$

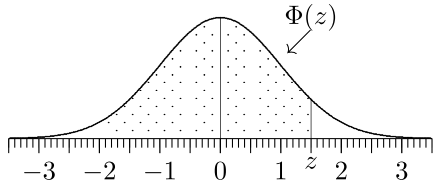

Table 1 CDF for Standard Normal.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0.6745) = 0.75$ $\Phi(1.6449) = 0.95$ $\Phi(2.3263) = 0.99$ $\Phi(3.0902) = 0.999$
$\Phi(1.2816) = 0.90$ $\Phi(1.9600) = 0.975$ $\Phi(2.5758) = 0.995$ $\Phi(3.2905) = 0.9995$

Distributions

Binomial	$\Pr[X = x]$	$=$	$\binom{n}{x} p^x q^{(n-x)},\ x = 0, 1, \ldots, n$	$\mu = np,\ \sigma^2 = npq$
Poisson	$\Pr[X = k]$	$=$	$\lambda^k e^{-\lambda}/k!,\ k = 0, 1, \ldots$	$\mu = \lambda,\ \sigma^2 = \lambda$
Poisson process	$\Pr[X = k \mid t]$	$=$	$(\alpha t)^k e^{-\alpha t}/k!,\ k = 0, 1, \ldots$	
Geometric	$\Pr[X = x]$	$=$	$pq^x,\ y = 0, 1, 2, \ldots$	$\mu = q/p,\ \sigma^2 = q/p^2$
Hypergeometric	$\Pr[X = x]$	$=$	$\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}},\ x = 0, 1, 2, \ldots$	$\mu = n\frac{M}{N}$
				$\sigma^2 = \frac{N-n}{N-1} n \frac{M}{N}(1 - \frac{M}{N})$
Exponential	$\Pr[X \le x]$	$=$	$1 - e^{-\lambda x},\ x \in (0, \infty)$	$\mu = 1/\lambda,\ \sigma^2 = 1/\lambda^2$
	$f(x)$	$=$	$\lambda e^{-\lambda x},\ x > 0;\ 0,\ x \le 0$	
Uniform	$\Pr[X \le x]$	$=$	$(x - a)/(b - a),\ x \in (a, b)$	$\mu = \frac{a+b}{2},\ \sigma^2 = \frac{(b-a)^2}{12}$
	$f(x)$	$=$	$\frac{1}{b-a},\ a<x<b;\ 0,\ x \notin (a, b)$	
Normal	$\Pr[X \le x]$	$=$	$\Phi(\frac{x-\mu}{\sigma}),\ x \in \mathbb{R}$	$\mu = \mu,\ \sigma^2 = \sigma^2$
	$f(x)$	$=$	$\frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2},\ x \in \mathbb{R}$	

$$P_k^n = \frac{n!}{(n-k)!} = \overbrace{(n)(n-1)\cdots(n-k+1)}^{k \text{ terms}}$$

$$\text{“}n \text{ choose } k\text{”: } \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n)(n-1)\cdots(n-k+1)}{(k)(k-1)\cdots(1)}$$

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$
$$\sum_{k=0}^\infty ar^k = a/(1-r) \qquad \sum_{k=0}^n ar^k = (a - ar^{n+1})/(1-r)$$
$$\Pr[X \le x] = F(x) = \int_{-\infty}^x f(t)\,dt \qquad f(x) = F'(x)$$
$$\mathsf{E}[g(X)] = \int g(x) f(x)\,dx \qquad \mathsf{E}[g(X)] = \sum_x g(x) P(x)$$
$$\mu = \mathsf{E}[X] = \int_{-\infty}^\infty x f(x)\,dx \qquad \sigma^2 = \mathsf{E}[(X-\mu)^2] = \mathsf{E}[X^2] - \mu^2$$

A **and** $B : A \cap B$ **not** $A : A^c$ A **or** $B : A \cup B$

$$\mathsf{P}[A|B] = \mathsf{P}[A \cap B]/\mathsf{P}[B] \qquad \mathsf{P}[B|A] = \mathsf{P}[A \cap B]/\mathsf{P}[A]$$
$$\mathsf{P}[A \cap B] = \mathsf{P}[A|B]\,\mathsf{P}[B] \qquad \mathsf{P}[A \cup B] = \mathsf{P}[A] + \mathsf{P}[B] - \mathsf{P}[A \cap B]$$

Independent $\Leftrightarrow \mathsf{P}[A \cap B] = \mathsf{P}[A]\,\mathsf{P}[B]$; Exclusive $\Leftrightarrow \mathsf{P}[A \cap B] = 0$

Bayes: $\mathsf{P}[A_i|B] = \frac{\mathsf{P}[B|A_i]\mathsf{P}[A_i]}{\mathsf{P}[B|A_1]\mathsf{P}[A_1] + \ldots + \mathsf{P}[B|A_k]\mathsf{P}[A_k]}$ $A_i \cap A_j = \emptyset, \quad \mathsf{P}[A_1] + \ldots + \mathsf{P}[A_k] = 1$

Sterling: $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

		Continuous	Discrete
Mean	$\mu = \mathsf{E}[X] =$	$\int_{-\infty}^{\infty} x\, f(x) dx$	$= \sum xp(x)$
Variance	$\sigma^2 = \mathsf{Var}[X] =$	$\int_{-\infty}^{\infty} (x-\mu)^2\, f(x) dx$	$= \sum (x-\mu)^2 p(x)$
		$= \mathsf{E}[X^2] - \mathsf{E}[X]^2$	$= \mathsf{E}[X^2] - \mathsf{E}[X]^2$
Expectation	$\mathsf{E}[g(X,Y)] =$	$\iint g(x,y)\, f(x,y) dx\, dy$	$= \sum_{x,y} g(x,y)\, p(x,y)$
Covariance	$\mathsf{Cov}[X,Y] =$	$\iint (x-\mu_x)(y-\mu_y)\, f(x,y) dx\, dy$	$= \sum_{x,y} (x-\mu_x)(y-\mu_y)\, p(x,y)$
		$= \mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y]$	$= \mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y]$
Correlation coefficient	$\rho[X,Y] =$	$\mathsf{Cov}[X,Y]/\sigma_X \sigma_Y$	
Conditional	$f(x\|y) =$	$f(x,y)/f_2(y)$	$= p(x,y)/p_2(y)$
Marginal	$f_1(x) =$	$\int_{-\infty}^{\infty} f(x,y) dy$	$= \sum_y p(x,y)$
	$f_2(y) =$	$\int_{-\infty}^{\infty} f(x,y) dx$	$= \sum_x p(x,y)$
Independent		$\Leftrightarrow f(x,y) = f_1(x)\, f_2(y)$	$\Leftrightarrow p(x,y) = p_1(x)\, p_2(y)$
		$\Leftrightarrow f(x\|y) = f_1(x)$	$\Leftrightarrow p(x\|y) = p_1(x)$
		$\Leftrightarrow f(y\|x) = f_2(y)$	$\Leftrightarrow p(y\|x) = p_2(y)$

Conf. Intervals	$\mu =$	$\bar{x} \pm z_{\alpha/2} S/\sqrt{n}$	Large Sample
		$= \bar{x} \pm t_{\alpha/2} S/\sqrt{n}$	Small Sample (normal)
	$p =$	$\hat{p} \pm z_{\alpha/2} \sqrt{\hat{p}\,\hat{q}\,/n}$	Population Proportion
Diff. of Means	$\mu_1 - \mu_2 =$	$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$	Large Samples
		$= (\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$	Small Samples
		$= \overline{x_1 - x_2} \pm z_{\alpha/2} s_\Delta/\sqrt{n}$	Matched Pairs, large n
		$= \overline{x_1 - x_2} \pm t_{\alpha/2} s_\Delta/\sqrt{n}$	Matched Pairs, small n, normal

Sample Mean	$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$	
Sample Variance	$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$	Deg fdm: $\nu = n - 1$
Regression slope	$b_1 = \hat{\beta}_1 = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}$	
Regression offset	$b_0 = \hat{\beta}_0 = \frac{\sum y_i - \hat{\beta}_1 \sum x_i}{n} = \bar{y} - \hat{\beta}_1 \bar{x}$	
Sum Square Error	$SSE = \sum (y_i - \hat{y}_i)^2$	
Estimate of variance	$\hat{\sigma^2} = \frac{SSE}{n-2}$	
TSS	$SST = \sum (y_i - \bar{y})^2$	
CD	$r^2 = 1 - \frac{SSE}{SST}$	
F-dist	$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$ F-dist with $m-1$,$n-1$ d.o.f.	