Sta-230: Probability **Summer 2020 I**

Key ideas for topic 14: Joint Distributions June 8

Joint distribution:
For the discrete case of two variables x, y the following are joint probabilities

$$\begin{aligned}\mathbf{P}(X = x, Y = x) &= P(x,y) \\ \mathbf{P}((X,Y) \in B) &= \sum_{(x,y)\in B} P(x,y),\end{aligned}$$

with

$$P(x,y) \geq 0, \quad \sum_x \sum_y P(x,y) = 1.$$

For the continuous case of two variables x, y the following is a joint probability density

$$\mathbf{P}(X \in dx, Y \in dy) = f(x,y)dx\,dy$$

with

$$f(x,y) \geq 0, \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dx\,dy = 1,$$

and

$$\mathbf{P}((X,Y) \in B) = \int_{(x,y)\in B} f(x,y)\,dx\,dy.$$

Independence:
For discrete distributions:

$$P(x,y) = P(X = x)P(Y = y).$$

For continuous distributions:

$$f(x,y) = f_X(x)f_Y(y).$$

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Key ideas for topic 15: Conditional and marginal distributions June 9

Joint and conditional probability for discrete random variables:
For the discrete case of two variables x, y the following relation between joint and conditional probabilities hold

$$\begin{aligned} P(X=x, Y=y) &= P(X=x)P(Y=y \mid X=x) \\ P(X=x \mid Y=y) &= \frac{P(X=x, Y=y)}{P(Y=y)} \\ P(Y=y \mid X=x) &= \frac{P(X=x, Y=y)}{P(X=x)}. \end{aligned}$$

Bayes rule

$$P(X=x \mid Y=y) = \frac{P(Y=y \mid X=x)P(X=x)}{P(Y+y)}.$$

Conditional probability and expectation:

$$\begin{aligned} P(Y \in B \mid X=x) &= \sum_{y \in B} P(Y=y \mid X=x) \\ \mathbb{E}(Y \mid X=x) &= \sum_{y} yP(Y=y \mid X=x). \end{aligned}$$

Marginals:

$$\begin{aligned} P(B) &= \sum_{x} P(B \mid X=x)P(X=x) \\ \mathbb{E}[Y] &= \sum_{x} \mathbb{E}(Y \mid X=x)P(X=x). \end{aligned}$$

Joint and conditional probability for continuous random variables:
For the continuous case of two variables x, y the following relation between joint and conditional probabilities hold

$$\begin{aligned} f(x,y) &= f_X(x) f(y \mid X = x) \\ f(y \mid X = x) &= \frac{f(x,y)}{f(x)} \\ f(x \mid Y = y) &= \frac{f(x,y)}{f(y)}. \end{aligned}$$

Bayes rule

$$f(X \mid Y = y) = \frac{f(y \mid X = x) f_X(x)}{f_Y(y)}.$$

Conditional probability and expectation:

$$\begin{aligned} P(Y \in B \mid X = x) &= \int_{y \in B} f_Y(y \mid X = x)\, dy \\ \mathbb{E}(Y \mid X = x) &= \int_y y f(y \mid X = x) dy. \end{aligned}$$

Marginals:

$$\begin{aligned} P(B) &= \int P(B \mid X = x) f_X(x) dx \\ \mathbb{E}[Y] &= \int \mathbb{E}(Y \mid X = x) f_X(x) dx. \end{aligned}$$

Key ideas for topic 16: Covariance and Correlation June 10

Covariance:
The covariance of two random variables X and Y is

$$\mathrm{Cov}(X,Y) = \mathbb{E}[(X-\mu_X)(Y-\mu_Y)], \quad \mu_X = \mathbb{E}[X], \quad \mu_Y = \mathbb{E}[Y],$$

and

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y).$$

Correlation:
The correlation between two random variables X and Y is

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}.$$

Cauchy-Schwarz-Bunyakovsky inequality:

$$|\mathbb{E}(XY)| \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$$

Cauchy-Schwarz tells the correlation is always between -1 and 1.

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Key ideas for topic 17: Tower rules for expectation and variance June 12

Adam's law:

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]].$$

The conditional variance:

$$\text{Var}(X|Y) = \mathbb{E}[(Y - \mathbb{E}[Y|X])^2|X]$$

is defined analogously as the usual variance, just everything conditional on X.

Eves' law:

$$\text{Var}(X) = \mathbb{E}[\text{Var}(X|Y)] + \text{Var}(\mathbb{E}[X|Y]).$$

Eve's law is a special case of more general phenomenon

$$\mathbb{E}[g(X,Y)] = \mathbb{E}[\mathbb{E}[g(X,Y)|Y]]$$