Sta-230: Probability **Summer I 2020**

Key ideas for topic 5: May 20 The Binomial Distribution

You should be able to think of distributions in two ways

(1) As a **sampling procedure or experiment**, for example flipping coins and recording the number of heads.

(2) As a **probability distribution function**

Bernoulli distribution: This is the coin flip distribution $X = \{0, 1\}$

$$\mathbf{P}(X = 1) = p, \quad \mathbf{P}(X = 0) = 1 - p, \quad \mathbf{P}(x; p) = p^x (1 - p)^{1-x}.$$

Binomial distribution: The random variable is the sum of n independent and identical (iid) trials of the Bernoulli distribution $X = \{0, 1, ..., n\}$ and

$$\mathbf{P}(x; p, n) = \binom{n}{x} p^x (1 - p)^{n-x}.$$

One can also think of this as an experiment of sampling with replacement. Consider a bin with 10^9 balls of which $10^9 \times p$ are blue and the remaining balls are red. Draw a ball from the bin and record the color of the ball and return the ball to the bin, replace this procedure n times. The distribution of the number of blue balls drawn is the Binomial distribution.

Cumulative distribution function: For the binomial distribution it is

$$F(z) = \mathbf{P}(z \leq x) = \sum_{x=0}^{z} \binom{n}{x} p^x (1 - p)^{n-x}.$$

In general for a discrete distribution

$$F(z) = \mathbf{P}(z \leq x) = \sum_{x=-\infty}^{z} \mathbf{P}(x).$$

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Key ideas for topic 5: May 21 The Normal Approximation

Normal approximation: The DeMoivre-Laplace central limit theorem says if n is large and p is not near 0 or 1 then the following distribution approximates the binomial

$$\mathbf{P}(x) \approx c e^{-(x-np)^2/(2np(1-p))}, \quad c = \frac{1}{2\pi np(1-p)}, \quad x = \{0, 1, ..., n\},$$

which we can rewrite as

$$\mathbf{P}(x) \approx \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}, \quad \mu = np, \quad \sigma^2 = np(1-p).$$

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Key ideas for topic 5: May 22 The Poisson distribution

Poisson distribution: A random variable Y is Poisson with parameter $\lambda > 0$ if

$$\mathbf{P}(Y = k; \lambda) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, ...$$

Poisson approximation: If n is large, p is small, and $np = \lambda$ then the binomial can be approximated by a Poisson distribution

$$\mathbf{P}(x) \approx \frac{e^{-\lambda}\lambda^x}{x!} \quad k = 0, 1, 2, ..., n.$$

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Key ideas for topic 6: May 26 Discrete distributions

Hypergeometric distribution: The hypergeometric can be considered as sampling without replacement. Consider a bin with N balls of which M are red, the remaining balls are blue. You draw n balls and record the number that are red as m. This procedure is sampling without replacement and the distribution of m is hypergeometric

$$\mathbf{P}(m; N, M, n) = \frac{\binom{n}{m}\binom{N-n}{M-m}}{\binom{N}{n}}.$$

Multinomial distribution: The multinomial distribution is a generalization of the binomial distribution. Consider K possible categories (instead of two) with the probability of drawing an observation for each category as p_k where $k = 1, ..., K$ with $p_k \geq 0$ and $\sum_{k=1}^{K} p_k = 1$. The probability of drawing $\{x_1, x_2, ..., x_{K-1}, x_k\}$ observations from each category assuming we draw n observations is

$$\mathbf{P}(\{x_1, ..., x_K\}; n, \{p_1, ..., p_k\}) = \frac{n!}{\prod_k x_k!} \prod_k p_k^{x_k}.$$

Geometric distribution: Consider two possible events $X = \{0, 1\}$ (say heads and tails) that are Bernoulli with parameter p and you want to write down the distribution of the number of times T you need to draw an observation until you observe a 1. This distribution is the geometric distribution

$$\mathbf{P}(t; p) = (1-p)^t p,$$

this is the probability of observing t zeros and then a one.

Infinite outcome spaces: The geometric and Poisson distributions are examples of discrete distributions with infinite outcome spaces, the number of possible outcomes is infinity. A property of both spaces is that there will be zero probability outcomes

$$\lim_{t\to\infty} \left[\mathbf{P}(t; p > 0) = (1-p)^t p\right] = 0, \quad \lim_{k\to\infty} \left[\mathbf{P}(k; \lambda < \infty) = \frac{e^{-\lambda}\lambda^k}{k!}\right] = 0.$$

To see why the second limit is zero note the following approximation

$$n! \approx C \left(\frac{n}{e}\right)^n \sqrt{n}$$

so

$$\frac{\lambda^k}{k!} \approx C^{-1} \left(\frac{e\lambda}{k}\right)^k, \quad \lim_{k\to\infty} \left[\left(\frac{e\lambda}{k}\right)^k\right] = 0.$$